

# One loop partition function from normal modes for $\mathcal{N} = 1$ supergravity in $\text{AdS}_3$

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**ABSTRACT:** With the recently discovered formula, which relates the off shell Euclidean one loop determinant to the on shell quantities such as normal and quasinormal modes in real spacetime, we work out the one loop partition function for  $\mathcal{N} = 1$  supergravity in  $\text{AdS}_3$  from scratch. In passing, we also provide the explicit expression for one loop determinant of a field with arbitrary spin in  $\text{AdS}_3$ . To achieve this, we firstly derive the determinant expression for the one loop partition function in question using the powerful decomposition technique, and then we construct the normal modes in a purely algebraic way by demonstrating that the space of normal modes falls into the representation of  $\text{SL}(2, R)$  Lie algebra associated to  $\text{AdS}_3$ . The whole procedure developed here turns out to be much simpler than the previous strategy.

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## 1. Introduction

AdS/CFT correspondence, as an explicit implementation of holographic principle, relates a  $d + 1$  dimensional gravitational theory to a  $d$  dimensional quantum field theory, where the black hole in the bulk is dual to the boundary field theory at a finite temperature. Most of checks and applications of this duality have restricted attention to the large  $N$  limit of boundary field theory, where gravity is treated as a classical object in the bulk. In order to capture a  $\frac{1}{N}$  correction to the boundary field theory by holography, one is required to calculate the one loop partition function for a quantum field propagating around the classical gravitational background mentioned above in the bulk.

Although it is technically challenging to compute the one loop partition function in a nontrivial gravitational background, there have been various methods developed to approximate it if the one loop partition function can not be determined explicitly. Among others, with the heat kernel method, the one loop partition can be calculated out explicitly for pure gravity in  $\text{AdS}_3$ [1]. Later on, such a heat kernel method is simplified

by using the group theoretical approach and the explicit expression is obtained for the one loop determinant of a field with arbitrary spin in  $\text{AdS}_3$  [2]. This is supposed to be not too surprising from both bulk and boundary perspectives in the context of  $\text{AdS}_3/\text{CFT}_2$  as there are sufficient symmetries on both sides to guarantee the integrability.

On the other hand, recently the authors in [3] have derived a new formula to evaluate the one loop bulk partition function in terms of normal or quasinormal modes, which arise as the solutions to the equation of motion for the field in question with the appropriate boundary conditions. This result is highly impressive because as alluded to above it relates the off shell object to the on shell quantities somehow. On the other hand, in the context of  $\text{AdS}/\text{CFT}$  correspondence the concept of normal and quasinormal modes is of interest in its own right. These modes correspond to the excitations on the boundary, as by the holographic recipe they arise as the poles of the retarded Green function of the dual operator.

Now with the above new formula for the one loop partition function, the task boils essentially down to the determination of normal or quasinormal modes, depending on whether the bulk has horizon or not. The approximate expressions for these modes offer a method for one to approximate the one loop partition function. In particular, the one loop partition function can be exactly evaluated through this new formula in those cases where the complete spectrum of normal or quasinormal modes can be nailed down. One of such cases is  $\text{AdS}_3$ . Actually as shown in [4] and [5], the spectrum of normal modes can be obtained as an infinite tower of descendants of the highest weight mode associated with the  $\text{SL}(2, R)$  Lie algebra of Killing fields for a scalar field in  $\text{AdS}_3$ .

The purpose of this paper is two fold. One is to construct the spectrum of normal modes in a similar way for any other higher spin field in  $\text{AdS}_3$  by demonstrating that the space of solutions to the equation of motion for the field in question falls into the representation of  $\text{SL}(2, R)$  Lie algebra. To achieve this, we shall employ the simplified computational strategy by sticking to the covariant derivative and introducing a pair of intrinsic tensor fields associated with the  $\text{SL}(2, R)$  symmetry [6]. The other is to demonstrate the advantage of the above new formula by working out the one loop partition function from normal modes for  $\mathcal{N} = 1$  supergravity in  $\text{AdS}_3$ , where instead of resorting to the Faddeev-Popov trick we shall derive the determinant expression for the one loop partition function in question by the decomposition technique, which turns out to be extremely efficient.

The rest of paper is organized as follows. In the next section we shall provide a brief review of the  $\text{SL}(2, R)$  Lie algebra of Killing fields in  $\text{AdS}_3$  as well as the two types of intrinsic tensor fields associated with the  $\text{SL}(2, R)$  quadratic Casimir operator. Then we describe the dynamics of  $\mathcal{N} = 1$  supergravity in  $\text{AdS}_3$  in Section 3, where in particular we show how the corresponding one loop partition function can be worked out in terms

of Laplace like determinants by the powerful decomposition technique. In Section 4, we will demonstrate how the spectrum of normal modes can be constructed in the algebraic manner as simple as possible by working explicitly on the case of gravitino field. In the subsequent section, as an application of the one loop determinant expression derived from normal modes for a field with arbitrary spin in  $\text{AdS}_3$ , we evaluate the one loop partition function for  $\mathcal{N} = 1$  supergravity, which turns out to be exactly the same as that obtained by the heat kernel method. We end up with some discussions in the final section and relegate various additional details to the four appendices.

Notation and conventions follow [7] unless specified otherwise.

## 2. $\text{AdS}_3$ and its $\text{SL}(2, R)$ symmetries

Start from the metric for  $\text{AdS}_3$ , i.e.,

$$ds^2 = -\cosh^2(\rho)dt^2 + \sinh^2(\rho)d\phi^2 + d\rho^2, \quad (2.1)$$

where  $0 \leq \rho \leq \infty$ ,  $-\infty \leq t \leq \infty$ , and  $-\pi \leq \phi \leq \pi$ . Whence the corresponding curvature reads

$$R_{abcd} = g_{ad}g_{bc} - g_{ac}g_{bd}, R_{ab} = -2g_{ab}, R = -6, \quad (2.2)$$

whereby  $\text{AdS}_3$  is a space of constant curvature, thus admits six Killing fields. We denote these six Killing fields by  $L_k$  and  $\bar{L}_k$  with  $k = 0, \pm 1$ . In particular,  $L_k$  is given by

$$\begin{aligned} L_0^a &= i\left(\frac{\partial}{\partial u}\right)^a, \\ L_{-1}^a &= ie^{-iu}\left[\frac{\cosh(2\rho)}{\sinh(2\rho)}\left(\frac{\partial}{\partial u}\right)^a - \frac{1}{\sinh(2\rho)}\left(\frac{\partial}{\partial v}\right)^a + \frac{i}{2}\left(\frac{\partial}{\partial \rho}\right)^a\right], \\ L_{+1}^a &= ie^{iu}\left[\frac{\cosh(2\rho)}{\sinh(2\rho)}\left(\frac{\partial}{\partial u}\right)^a - \frac{1}{\sinh(2\rho)}\left(\frac{\partial}{\partial v}\right)^a - \frac{i}{2}\left(\frac{\partial}{\partial \rho}\right)^a\right], \end{aligned} \quad (2.3)$$

where we have employed another coordinate system, namely  $\{\rho, u = t + \phi, v = t - \phi\}$ .  $\bar{L}_k$  can be similarly defined as (2.6) except switch  $u$  and  $v$  therein. Then it can be shown that their Lie commutators satisfy two sets of the  $\text{SL}(2, R)$  Lie algebra, i.e.,

$$[L_0, L_{\pm 1}] = \mp L_{\pm 1}, [L_{+1}, L_{-1}] = 2L_0, [\bar{L}_0, \bar{L}_{\pm 1}] = \mp \bar{L}_{\pm 1}, [\bar{L}_{+1}, \bar{L}_{-1}] = 2\bar{L}_0, [L_k, \bar{L}_l] = 0. \quad (2.4)$$

Note that the Lie derivative observes  $[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X, Y]}$  and  $\mathcal{L}_{\alpha X} = \alpha \mathcal{L}_X$  for the arbitrary Killing vector fields  $X$  and  $Y$  with the arbitrary constant  $\alpha$ . Thus the above

Lie algebra can be naturally represented by the Lie derivative. In particular, the quadratic Casimir operators can be realized by the Lie derivative as

$$\mathcal{L}^2 = \mathcal{L}_{L_0} \mathcal{L}_{L_0} - \frac{1}{2}(\mathcal{L}_{L_{+1}} \mathcal{L}_{L_{-1}} + \mathcal{L}_{L_{-1}} \mathcal{L}_{L_{+1}}), \bar{\mathcal{L}}^2 = \mathcal{L}_{\bar{L}_0} \mathcal{L}_{\bar{L}_0} - \frac{1}{2}(\mathcal{L}_{\bar{L}_{+1}} \mathcal{L}_{\bar{L}_{-1}} + \mathcal{L}_{\bar{L}_{-1}} \mathcal{L}_{\bar{L}_{+1}}), \quad (2.5)$$

which commute with both  $\mathcal{L}_{L_k}$  and  $\mathcal{L}_{\bar{L}_k}$ .

As advocated in [6], it turns out to be convenient for tensor and spinor analysis related to the above quadratic  $\text{SL}(2, R)$  Casimir if one constructs the two types of auxiliary tensor fields as follows<sup>1</sup>

$$H^{ab} = L_0^a L_0^b - \frac{1}{2}(L_{+1}^a L_{-1}^b + L_{-1}^a L_{+1}^b), \bar{H}^{ab} = \bar{L}_0^a \bar{L}_0^b - \frac{1}{2}(\bar{L}_{+1}^a \bar{L}_{-1}^b + \bar{L}_{-1}^a \bar{L}_{+1}^b), \quad (2.6)$$

and

$$\begin{aligned} Z_{abc} &= L_{0a} \nabla_b L_{0c} - \frac{1}{2}(L_{+1a} \nabla_b L_{-1c} + L_{-1a} \nabla_b L_{+1c}), \\ \bar{Z}_{abc} &= \bar{L}_{0a} \nabla_b \bar{L}_{0c} - \frac{1}{2}(\bar{L}_{+1a} \nabla_b \bar{L}_{-1c} + \bar{L}_{-1a} \nabla_b \bar{L}_{+1c}). \end{aligned} \quad (2.7)$$

The essential reason is that they possess the following nice properties, i.e.,

$$H^{ab} = \bar{H}^{ab} = \frac{1}{4}g^{ab}, \quad (2.8)$$

and

$$Z_{abc} = \frac{1}{4}\epsilon_{abc}, \bar{Z}_{abc} = -\frac{1}{4}\epsilon_{abc}, \quad (2.9)$$

where  $g$  and  $\epsilon$  are the metric and volume element of our  $\text{AdS}_3$  respectively.

### 3. $\mathcal{N} = 1$ supergravity and its determinant expression for one loop partition function

#### 3.1 Graviton

Let us start to consider the dynamics of graviton  $h_{ab}$  on top of  $\text{AdS}_3$  by focusing on the perturbative expansion of the Einstein-Hilbert action with a negative cosmological constant  $\Lambda = -1$ , i.e.,

$$S = \frac{1}{16\pi G} \int d^3x \sqrt{-g} (R - 2\Lambda) \quad (3.1)$$

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<sup>1</sup>Although the whole construction is built up for the BTZ black hole in [6], it can be readily translated into  $\text{AdS}_3$  because  $\text{AdS}_3$  is related to the BTZ black hole in [6] by change of coordinates as  $\{t \rightarrow i\phi, \phi \rightarrow it, \rho \rightarrow \rho\}$ .

to the quadratic order around the background. Note that the zero order term is irrelevant to our purpose and the first order one vanishes due to the fact that the background metric arises as a solution to the above action. So what we need to know is merely the quadratic term, which is given by

$$S_h = -\frac{1}{64\pi G} \int d^3x \sqrt{-g} h^{ab} (\nabla_a \nabla_c h^c_b + \nabla_b \nabla_c h^c_a - \nabla^c \nabla_c h_{ab} - \nabla_a \nabla_b h - 2h_{ab} - g_{ab} \nabla^c \nabla^d h_{cd} + g_{ab} \nabla^c \nabla_c h), \quad (3.2)$$

whereby the equation of motion can be obtained by the variational principle as

$$\nabla_a \nabla_c h^c_b + \nabla_b \nabla_c h^c_a - \nabla^c \nabla_c h_{ab} - \nabla_a \nabla_b h - 2h_{ab} - g_{ab} \nabla^c \nabla^d h_{cd} + g_{ab} \nabla^c \nabla_c h = 0 \quad (3.3)$$

with  $h$  the trace of  $h_{ab}$ . Note that the dynamics of graviton has the gauge freedom inherited from the diffeomorphism invariance of full theory. So with the transverse traceless gauge conditions  $\nabla^a h_{ab} = 0$  and  $h = 0$ , the equation of motion can be reduced to

$$(\nabla^c \nabla_c + 2)h_{ab} = 0. \quad (3.4)$$

Conventionally one uses the Faddeev-Popov trick to work out the determinant expression for one loop partition function. Here we would like to employ the alternative approach, which is proven to be more efficient[8]. To achieve this, let us firstly decompose the metric as follows

$$h_{ab} = h_{ab}^T + \frac{1}{3}g_{ab}\alpha + \nabla_a \xi_b + \nabla_b \xi_a, \quad (3.5)$$

where  $h_{ab}^T$  satisfies the transverse traceless condition, and the last two terms come from the diffeomorphism contribution<sup>2</sup>. With this decomposition, we have

$$\begin{aligned} \int \mathcal{D}h_{ab} e^{S_h} &= \int Z_{ghost} \mathcal{D}h_{ab}^T \mathcal{D}\alpha \mathcal{D}\xi_a e^{-\frac{1}{64\pi G} \int d^3x \sqrt{-g} h^{T ab} (-\nabla^c \nabla_c - 2) h_{ab}^T + \frac{2}{9} \alpha (\nabla^c \nabla_c - 3) \alpha} \\ &= \int Z_{ghost} \mathcal{D}\xi_a [\det_2(-\nabla^c \nabla_c - 2)]^{-\frac{1}{2}} [\det_0(\nabla^c \nabla_c - 3)]^{-\frac{1}{2}}, \end{aligned} \quad (3.6)$$

which implies that the one loop partition function is given by throwing away the whole volume of diffeomorphism as

$$Z_{graviton} = Z_{ghost} [\det_2(-\nabla^c \nabla_c - 2)]^{-\frac{1}{2}} [\det_0(\nabla^c \nabla_c - 3)]^{-\frac{1}{2}}, \quad (3.7)$$

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<sup>2</sup>It is necessary to preserve the degrees of freedom when one performs such a decomposition as we do here, namely the degrees of freedom is six on both sides. Otherwise, one would run into a wrong determinant expression for one loop partition function by using such an alternative approach at the end of day. Put it another way, this sort of decomposition can be regarded as sort of change of variables, which is well defined only with the degrees of freedom preserved. Otherwise, the corresponding Jacobian would be degenerate somehow.

where  $Z_{ghost}$  arises as the Jacobian corresponding to change of variables from  $h_{ab}$  to  $\{h_{ab}^T, \alpha, \xi_a\}$ . Obviously, in order to nail down the one loop partition function for graviton, we are required to work out the specific expression for  $Z_{ghost}$ , which can be accomplished in Appendix B as

$$Z_{ghost} = [\det_1(-\nabla^a \nabla_a + 2)]^{\frac{1}{2}} [\det_0(-\nabla^a \nabla_a + 3)]^{\frac{1}{2}}. \quad (3.8)$$

Thus the determinant expression for one loop partition function of graviton reads

$$Z_{graviton} = [\det_2(-\nabla^c \nabla_c - 2)]^{-\frac{1}{2}} [\det_1(-\nabla^a \nabla_a + 2)]^{\frac{1}{2}}. \quad (3.9)$$

### 3.2 Gravitino

Start from the gravitino action

$$S_\varphi = - \int d^3x \sqrt{-g} \bar{\varphi}_a (\gamma^{abc} \nabla_b \varphi_c + \hat{m} \gamma^{ab} \varphi_b) = - \int d^3x \sqrt{-g} \bar{\varphi}_a \epsilon^{abc} (\nabla_b \varphi_c - \hat{m} \gamma_b \varphi_c). \quad (3.10)$$

Here  $\hat{m}^2 = \frac{1}{4}$ ,  $\bar{\varphi} = \varphi^\dagger \gamma^0$ ,  $\gamma^{ab} = \gamma^{[a} \gamma^{b]}$ , and  $\gamma^{abc} = \gamma^{[a} \gamma^b \gamma^{c]}$  with  $\gamma^a = e_I^a \gamma^I$ , where  $e_I^a$  form a set of orthogonal normal vector bases, and Gamma matrices are defined as  $\gamma^{(I} \gamma^{J)} = \eta^{IJ}$  with the requirement  $\gamma^0 \gamma^{I\dagger} \gamma^0 = \gamma^I$ . Then the equation of motion for gravitino field can be obtained by the variational principle as

$$\gamma^{abc} \nabla_b \varphi_c + \hat{m} \gamma^{ab} \varphi_b = 0. \quad (3.11)$$

Note that the above action is invariant under the gauge transformation  $\delta \varphi_a = \nabla_a \zeta - \hat{m} \gamma_a \zeta$  as

$$\begin{aligned} \epsilon^{abc} (\nabla_b \delta \varphi_c - \hat{m} \gamma_b \delta \varphi_c) &= \epsilon^{abc} (\nabla_{[b} \nabla_{c]} \zeta + \hat{m}^2 \gamma_{bc} \zeta) = \epsilon^{abc} \left( \frac{1}{8} R_{bcde} \gamma^{de} \zeta + \hat{m}^2 \gamma_{bc} \zeta \right) \\ &= \epsilon^{abc} \left( -\frac{1}{4} + \hat{m}^2 \right) \gamma_{bc} \zeta = 0. \end{aligned} \quad (3.12)$$

So we can always impose the covariant gauge condition on gravitino field as  $\gamma^a \varphi_a = \nabla^a \varphi_a = 0$ , which gives rise to

$$\gamma^{abc} \nabla_b \varphi_c = \gamma^a \gamma^b \gamma^c \nabla_b \varphi_c + (-g^{ab} \gamma^c + g^{ca} \gamma^b - g^{bc} \gamma^a) \nabla_b \varphi_c = \gamma^b \nabla_b \varphi^a, \quad (3.13)$$

and

$$\hat{m} \gamma^{ab} \varphi_b = \hat{m} (\gamma^a \gamma^b - g^{ab}) \varphi_b = -\hat{m} \varphi^a. \quad (3.14)$$

Thus the equation of motion can be simplified as

$$(\gamma^b \nabla_b - \hat{m}) \varphi_a = 0. \quad (3.15)$$

On the other hand, in order to obtain the determinant expression for one loop partition function of gravitino field, let us firstly decompose the gravitino field as

$$\varphi_a = \varphi_a^T + \frac{1}{3}\gamma_a\chi + \nabla_a\zeta - \hat{m}\gamma_a\zeta, \quad (3.16)$$

where  $\varphi_a^T$  satisfies the covariant gauge condition. With such a decomposition, we have

$$\begin{aligned} \int \mathcal{D}\varphi_a e^{S_\varphi} &= \int Z_{ghost} \mathcal{D}\varphi_a^T \mathcal{D}\chi \mathcal{D}\zeta e^{-\int d^3x \sqrt{-g} \bar{\varphi}_b^T (\gamma^a \nabla_a - \hat{m}) \varphi^{Tb} - \frac{2}{9} \bar{\chi} (\gamma^a \nabla_a + 3\hat{m}) \chi} \\ &= \int Z_{ghost} d\zeta \det_{\frac{3}{2}}(\gamma^a \nabla_a - \hat{m}) \det_{\frac{1}{2}}(\gamma^a \nabla_a + 3\hat{m}), \end{aligned} \quad (3.17)$$

where  $Z_{ghost}$  is obtained in Appendix B as

$$Z_{ghost} = [\det_{\frac{1}{2}}(-\nabla^a \nabla_a + \frac{3}{4})]^{-1}. \quad (3.18)$$

Thus the one loop partition function for gravitino field is given by

$$\begin{aligned} Z_{gravitino} &= \det_{\frac{3}{2}}(\gamma^a \nabla_a - \hat{m}) \det_{\frac{1}{2}}(\gamma^a \nabla_a + 3\hat{m}) [\det_{\frac{1}{2}}(-\nabla^a \nabla_a + \frac{3}{4})]^{-1} \\ &= [\det_{\frac{3}{2}}(-\nabla^a \nabla_a - \frac{9}{4})^{\frac{1}{2}} [\det_{\frac{1}{2}}(-\nabla^a \nabla_a + \frac{3}{4})]^{-\frac{1}{2}}]. \end{aligned} \quad (3.19)$$

Here we have used the following facts, i.e.,

$$\det(\gamma^a \nabla_a + \hat{m}) = \det(\gamma^a \nabla_a - \hat{m}), \quad (3.20)$$

and

$$\det_s(\gamma^a \nabla_a + \hat{m}) \det_s(\gamma^a \nabla_a - \hat{m}) = \det_s[(\gamma^a \nabla_a + \hat{m})(\gamma^a \nabla_a - \hat{m})] = \det_s(\nabla^a \nabla_a - \hat{m}^2 + s + 1), \quad (3.21)$$

where the proof of the former can be found in the Appendix C, and the latter will be demonstrated in the next section.

## 4. Normal modes for gravitino and other fields

In this section, we will construct the spectrum of normal modes for gravitino field around our AdS<sub>3</sub> in an algebraic way. In addition, for later computation, we also include the relevant result for other fields in the end.

Let us get started by acting on both sides of the equation of motion for gravitino field (3.15) with  $\gamma^c \nabla_c + \hat{m}$ , which gives rise to

$$\begin{aligned} 0 &= (\gamma^c \nabla_c + \hat{m})(\gamma^b \nabla_b - \hat{m})\varphi_a = (\gamma^c \gamma^b \nabla_c \nabla_b - \hat{m}^2)\varphi_a \\ &= (g^{cb} \nabla_c \nabla_b + \gamma^{cb} \nabla_c \nabla_b - \hat{m}^2)\varphi_a = (\nabla^b \nabla_b + \gamma^{cb} \nabla_{[c} \nabla_{b]} - \hat{m}^2)\varphi_a. \end{aligned} \quad (4.1)$$



Note that

$$\nabla_{[c}\nabla_{b]}\varphi_a = \frac{1}{8}R_{cbde}\gamma^{de}\varphi_a + \frac{1}{2}R_{cba}{}^d\varphi_d. \quad (4.2)$$

Thus we can further have

$$0 = (\nabla^b\nabla_b + \frac{1}{8}R_{cbde}\gamma^{cb}\gamma^{de} - \hat{m}^2)\varphi_a + \frac{1}{2}\gamma^{cb}R_{cba}{}^d\varphi_d, \quad (4.3)$$

which can be reduced to

$$(\nabla^b\nabla_b + \frac{5}{2} - \hat{m}^2)\varphi_a = 0 \quad (4.4)$$

on our  $\text{AdS}_3$ .

On the other hand, by the definition of Lie derivative acting on  $\varphi_a$ , i.e.,

$$\mathcal{L}_\xi\varphi_a = \xi^b\nabla_b\varphi_a - \frac{1}{4}\gamma^{cd}\psi_a\nabla_d\xi_c + \varphi_b\nabla_a\xi^b, \quad (4.5)$$

we have

$$\begin{aligned} \mathcal{L}_X\mathcal{L}_Y\varphi_f &= X^a\nabla_a\mathcal{L}_Y\varphi_f - \frac{1}{4}\gamma^{ab}\mathcal{L}_Y\varphi_f\nabla_bX_a + \mathcal{L}_Y\varphi_a\nabla_fX^a \\ &= X^a\nabla_a(Y^c\nabla_c\varphi_f - \frac{1}{4}\gamma^{cd}\varphi_f\nabla_dY_c + \varphi_c\nabla_fY^c) \\ &\quad - \frac{1}{4}\gamma^{ab}(Y^c\nabla_c\varphi_f - \frac{1}{4}\gamma^{cd}\varphi_f\nabla_dY_c + \varphi_c\nabla_fY^c)\nabla_bX_a \\ &\quad + (Y^c\nabla_c\varphi_a - \frac{1}{4}\gamma^{cd}\varphi_a\nabla_dY_c + \varphi_c\nabla_aY^c)\nabla_fX^a \\ &= X^a(\nabla_aY^c)\nabla_c\varphi_f + X^aY^c\nabla_a\nabla_c\varphi_f - \frac{1}{4}\gamma^{cd}(\nabla_a\varphi_f)X^a\nabla_dY_c - \frac{1}{4}\gamma^{cd}\varphi_fX^a\nabla_a\nabla_dY_c \\ &\quad + (\nabla_a\varphi_c)X^a\nabla_fY^c + \varphi_cX^a\nabla_a\nabla_fY^c \\ &\quad - \frac{1}{4}\gamma^{ab}(\nabla_c\varphi_f)Y^c\nabla_bX_a + \frac{1}{16}\gamma^{ab}\gamma^{cd}\varphi_f(\nabla_dY_c)\nabla_bX_a - \frac{1}{4}\gamma^{ab}\varphi_c(\nabla_fY^c)\nabla_bX_a \\ &\quad + (\nabla_c\varphi_a)Y^c\nabla_fX^a - \frac{1}{4}\gamma^{cd}\varphi_a(\nabla_dY_c)\nabla_fX^a + \varphi_c(\nabla_aY^c)\nabla_fX^a \\ &= X^a(\nabla_aY^c)\nabla_c\varphi_f + X^aY^c\nabla_a\nabla_c\varphi_f - \frac{1}{4}\gamma^{cd}(\nabla_a\varphi_f)X^a\nabla_dY_c - \frac{1}{4}\gamma^{cd}\varphi_fX^a\nabla_a\nabla_dY_c \\ &\quad + (\nabla_a\varphi_c)X^a\nabla_fY^c + \varphi_cX^a\nabla_a\nabla_fY^c - \frac{1}{4}\gamma^{ab}(\nabla_c\varphi_f)Y^c\nabla_bX_a \\ &\quad + \frac{1}{16}\gamma^{ab}\gamma^{cd}\varphi_f[\nabla_d(Y_c\nabla_bX_a) - Y_c\nabla_d\nabla_bX_a] - \frac{1}{4}\gamma^{ab}\varphi_c[\nabla_f(Y^c\nabla_bX_a) - Y^c\nabla_f\nabla_bX_a] \\ &\quad + (\nabla_c\varphi_a)Y^c\nabla_fX^a - \frac{1}{4}\gamma^{cd}\varphi_a[\nabla_d(Y_c\nabla_fX^a) - Y_c\nabla_d\nabla_fX^a] \\ &\quad + \varphi_c[\nabla_a(Y^c\nabla_fX^a) - Y^c\nabla_a\nabla_fX^a]. \end{aligned} \quad (4.6)$$

Whence it is not hard to show

$$\begin{aligned}
\mathcal{L}^2 \varphi_f &= Z^a{}_c \nabla_c \varphi_f + H^{ac} \nabla_a \nabla_c \varphi_f - \frac{1}{4} Z^a{}_{dc} \gamma^{cd} \nabla_a \varphi_f - \frac{1}{4} \gamma^{cd} \varphi_f R_{cdae} H^{ae} \\
&\quad + Z^a{}_f{}^c \nabla_a \varphi_c + \varphi_c R^c{}_{fad} H^{ad} - \frac{1}{4} Z^c{}_{ba} \gamma^{ab} \nabla_c \varphi_f \\
&\quad + \frac{1}{16} \gamma^{ab} \gamma^{cd} \varphi_f (\nabla_d Z_{cba} - R_{abde} H_c{}^e) - \frac{1}{4} \gamma^{ab} \varphi_c (\nabla_f Z^c{}_{ba} - R_{abfd} H^{cd}) \\
&\quad + Z^c{}_f{}^a \nabla_c \varphi_a - \frac{1}{4} \gamma^{cd} \varphi_a (\nabla_d Z_c{}^f{}^a - R^a{}_{fde} H_c{}^e) + \varphi_c (\nabla_a Z^c{}_f{}^a - R^a{}_{fad} H^{cd}) \\
&= \frac{1}{4} (\nabla^a \nabla_a \varphi_f - \frac{1}{4} \epsilon^a{}_{dc} \gamma^{cd} \nabla_a \varphi_f + \epsilon^a{}_f{}^c \nabla_a \varphi_c - \frac{1}{4} \epsilon^c{}_{ba} \gamma^{ab} \nabla_c \varphi_f \\
&\quad - \frac{1}{16} R_{abdc} \gamma^{ab} \gamma^{cd} \varphi_f + \frac{1}{4} R_{abfd} \gamma^{ab} \varphi^d + \epsilon^c{}_f{}^a \nabla_c \varphi_a + \frac{1}{4} R^a{}_{fdc} \gamma^{cd} \varphi_a - R^a{}_{fad} \varphi^d) \\
&= \frac{1}{4} (\nabla^a \nabla_a \varphi_f + \frac{1}{2} \epsilon^a{}_{cd} \gamma^{cd} \nabla_a \varphi_f - 2 \epsilon_f{}^{ac} \nabla_a \varphi_c \\
&\quad + \frac{1}{16} R_{abcd} \gamma^{ab} \gamma^{cd} \varphi_f + \frac{1}{2} R_{abfd} \gamma^{ab} \varphi^d - R_{fd} \varphi^d), \tag{4.7}
\end{aligned}$$

where we have used the identity  $\nabla_a \nabla_b \xi_c = R_{cbad} \xi^d$  satisfied by any Killing vector field  $\xi$ . Now substitute (4.3) into the above equation, we can obtain

$$\begin{aligned}
\mathcal{L}^2 \varphi_f &= \frac{1}{4} (\hat{m}^2 \varphi_f + \frac{1}{2} \epsilon^a{}_{cd} \gamma^{cd} \nabla_a \varphi_f - 2 \epsilon_f{}^{ac} \nabla_a \varphi_c - \frac{1}{16} R_{abcd} \gamma^{ab} \gamma^{cd} \varphi_f - R_{fd} \varphi^d) \\
&= \frac{1}{4} (\hat{m}^2 \varphi_f - \gamma^a \nabla_a \varphi_f - 2 \gamma_f{}^{ac} \nabla_a \varphi_c - \frac{1}{16} R_{abcd} \gamma^{ab} \gamma^{cd} \varphi_f - R_{fd} \varphi^d) \\
&= \frac{1}{4} (\hat{m}^2 \varphi_f - 3 \gamma^a \nabla_a \varphi_f - \frac{1}{16} R_{abcd} \gamma^{ab} \gamma^{cd} \varphi_f - R_{fd} \varphi^d) \\
&= \frac{1}{4} (\hat{m}^2 \varphi_f - 3 \hat{m} \varphi_f - \frac{1}{16} R_{abcd} \gamma^{ab} \gamma^{cd} \varphi_f - R_{fd} \varphi^d) \\
&= \frac{1}{4} (\hat{m}^2 - 3 \hat{m} + \frac{5}{4}) \varphi_f. \tag{4.8}
\end{aligned}$$

Similarly, one can have

$$\bar{\mathcal{L}}^2 \varphi_f = \frac{1}{4} (\hat{m}^2 + 3 \hat{m} + \frac{5}{4}) \varphi_f. \tag{4.9}$$

Together with the fact that the Lie derivative via Killing vector fields commutes with the covariant derivative and Gamma matrices as well, the above result implies that the space of solutions to the equation of motion for gravitino field forms one representation of  $SL(2, R)$  Lie algebra, which is characterized by the value of the Casimir. Actually the same situation occurs for other fields. Moreover, the whole result can be cast into a

uniform pattern. Namely as to a field  $\Phi$  with spin  $s$  satisfying the Laplace like equation<sup>3</sup>

$$(\nabla^a \nabla_a - \hat{m}^2 + s + 1)\Phi = 0, \quad (4.10)$$

one can have

$$\mathcal{L}^2 \Phi = \frac{\hat{m}^2 - 2s\hat{m} + s^2 - 1}{4} \Phi, \bar{\mathcal{L}}^2 \Phi = \frac{\hat{m}^2 + 2s\hat{m} + s^2 - 1}{4} \Phi. \quad (4.11)$$

Then we can construct the corresponding spectrum of normal modes in an algebraic way by starting from the highest weight mode which obeys the condition as follows

$$\mathcal{L}_{L+1} \Phi^{(0,0)} = 0, \mathcal{L}_{\bar{L}+1} \Phi^{(0,0)} = 0, \mathcal{L}_{L_0} \Phi^{(0,0)}_+ = w_+ \Phi^{(0,0)}, \mathcal{L}_{\bar{L}_0} \Phi^{(0,0)} = w_- \Phi^{(0,0)} \quad (4.12)$$

where  $\Phi^{(0,0)}$  can be always expressed as

$$\Phi^{(0)} = e^{-iw_+ u + iw_- v} \Phi^{(0,0)}(\rho) \quad (4.13)$$

with  $\mathcal{L}_{L_0} \Phi^{(0,0)}(\rho) = \mathcal{L}_{\bar{L}_0} \Phi^{(0,0)}(\rho) = 0$ . Then the resultant normal modes can be obtained as the infinite tower of descendant modes, i.e.,

$$\Phi^{(p,q)} = (\mathcal{L}_{\bar{L}-1})^p (\mathcal{L}_{L-1})^q \Phi^{(0,0)} \quad (4.14)$$

with  $p, q = 0, 1, 2, \dots$ . Note that  $\bar{L}_0 + L_0$  and  $\bar{L}_0 - L_0$  represent the energy and angular momentum respectively. So the corresponding frequencies and angular momentum can be calculated out as

$$\omega^{(p,q)} = w_- + w_+ + p + q, j^{(p,q)} = w_- - w_+ + p - q. \quad (4.15)$$

Using the commutation relation (2.5), one can show that the conformal weight is given by

$$w_+ = \frac{1 \pm (\hat{m} - s)}{2}, w_- = \frac{1 \pm (\hat{m} + s)}{2}. \quad (4.16)$$

As expected, the whole result is in good agreement with the prediction from the two dimensional CFT, where the conformal dimension and spin of dual primary operator to the bulk field are given by<sup>4</sup>

$$\Delta = w_+ + w_- = 1 \pm \hat{m}, s = |w_+ - w_-|. \quad (4.17)$$

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<sup>3</sup>Although it is explicitly proven to be true for gravitino field in this section and for other fields with spin no greater than 2 in [6] for example, which turns out to be sufficient for our later calculation, we are believed that this uniform pattern is expected to be also true for fields with any other spin from both bulk and boundary points of view.

<sup>4</sup>Note that the relevant result is symmetric with respect to  $\hat{m}$  and  $-\hat{m}$ . So in what follows, we shall work solely on one of them, say the case in which the value of  $\hat{m}$  is positive. On the other hand, to have the modes normalizable as it stands, both  $w_+$  and  $w_-$  are required to be non-negative, which is reasonable also from the boundary point of view, since otherwise the dual operator would possess the correlation functions increasing with the distance.

## 5. Determinant of Laplace like operator from normal modes and one loop partition function for $\mathcal{N} = 1$ supergravity

This section is to show how the determinant of Laplace like operator can be evaluated through the spectrum of normal modes we obtained in the last section and work out the one loop partition function for  $\mathcal{N} = 1$  supergravity as an application.

To proceed, let us recall the fact that the Euclidean determinant of Laplace like operator in Eq.(4.10), i.e.,

$$D_s(\Delta) = \det_s(\nabla^a \nabla_a - \hat{m}^2 + s + 1) \quad (5.1)$$

depends on the boundary condition, which is specified in terms of the conformal dimension  $\Delta$  by the holographic recipe. Then as shown in [3], by continuing to complex  $\Delta$  and matching poles as well as zeros, one can determine  $D_s(\Delta)$  through the spectrum of normal modes as follows

$$\begin{aligned} D_s(\Delta) &= e^{\text{Pol}(\Delta)} \prod_{p,q} (2 \sinh \frac{\omega^{(p,q)}}{2T})^{2n} = e^{\text{Pol}(\Delta)} \prod_{p,q} \left( \frac{1 - e^{-\frac{\omega^{(p,q)}}{T}}}{e^{-\frac{\omega^{(p,q)}}{2T}}} \right)^{2n} \\ &= e^{\text{Pol}(\Delta)} \prod_{k \geq 0} \left( \frac{1 - e^{-\frac{\Delta+k}{T}}}{e^{-\frac{\Delta+k}{2T}}} \right)^{2n(k+1)} \end{aligned} \quad (5.2)$$

for bosonic fields and

$$\begin{aligned} D_s(\Delta) &= e^{\text{Pol}(\Delta)} \prod_{p,q} (2 \cosh \frac{\omega^{(p,q)}}{2T})^{2n} = e^{\text{Pol}(\Delta)} \prod_{p,q} \left( \frac{1 + e^{-\frac{\omega^{(p,q)}}{T}}}{e^{-\frac{\omega^{(p,q)}}{2T}}} \right)^{2n} \\ &= e^{\text{Pol}(\Delta)} \prod_{k \geq 0} \left( \frac{1 + e^{-\frac{\Delta+k}{T}}}{e^{-\frac{\Delta+k}{2T}}} \right)^{2n(k+1)} \end{aligned} \quad (5.3)$$

for fermionic fields, where  $T$  denotes the temperature for the thermal  $\text{AdS}_3$ ,  $n$  is the number of degrees of freedom of integrated fields, and  $\text{Pol}(\Delta)$  represents the polynomial function of  $\Delta$ <sup>5</sup>. Furthermore, by absorbing various terms legitimately into  $\text{Pol}(\Delta)$ , we end up with

$$D_s(\Delta) = e^{\text{Pol}(\Delta)} \prod_{k \geq 0} (1 - q^{\Delta+k})^{2n(k+1)} \quad (5.4)$$

for bosonic fields and

$$D_s(\Delta) = e^{\text{Pol}(\Delta)} \prod_{k \geq 0} (1 + q^{\Delta+k})^{2n(k+1)} \quad (5.5)$$

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<sup>5</sup>The specific form of this polynomial function is irrelevant to our current purpose, though it can be fixed by taking the large  $\Delta$  limit indeed[3].

for fermionic fields, where

$$q = e^{2\pi i\tau} \quad (5.6)$$

with  $\tau = \frac{i}{2\pi T}$ .

Now it is time for us to evaluate the one loop determinant for  $\mathcal{N} = 1$  supergravity. To proceed, let us firstly read off the conformal dimension for spin 2 and spin 1 field from (3.9) as

$$\Delta_2 = 2, \Delta_1 = 3, \quad (5.7)$$

where (4.10) and (4.17) are used. Furthermore, by taking into account the fact that the number of degrees of freedom is two for both integrated spin 2 and spin 1 fields, we have

$$\begin{aligned} Z_{graviton} &= \prod_{k \geq 0} (1 - q^{2+k})^{-2(k+1)} \prod_{k' \geq 0} (1 - q^{3+k'})^{2(k'+1)} \\ &= \prod_{k \geq 0} (1 - q^{2+k})^{-2(k+1)} \prod_{k' \geq 1} (1 - q^{2+k'})^{2k'} \\ &= \prod_{k \geq 0} \frac{1}{(1 - q^{2+k})^2}, \end{aligned} \quad (5.8)$$

where we have thrown away the irrelevant polynomial term  $e^{\text{Pol}(\Delta)}$ .

Similarly, we can read off the conformal dimension for spin  $\frac{3}{2}$  and spin  $\frac{1}{2}$  field from (3.19) as

$$\Delta_{\frac{3}{2}} = \frac{3}{2}, \Delta_{\frac{1}{2}} = \frac{5}{2}, \quad (5.9)$$

whereby the one loop partition function for gravitino field is given by

$$\begin{aligned} Z_{gravitino} &= \prod_{k \geq 0} (1 + q^{\frac{3}{2}+k})^{2(k+1)} \prod_{k' \geq 0} (1 + q^{\frac{5}{2}+k'})^{-2(k'+1)} \\ &= \prod_{k \geq 0} (1 + q^{\frac{3}{2}+k})^{2(k+1)} \prod_{k' \geq 1} (1 + q^{\frac{3}{2}+k'})^{-2k'} \\ &= \prod_{k \geq 0} (1 + q^{\frac{3}{2}+k})^2. \end{aligned} \quad (5.10)$$

Combine (5.8) with (5.10), we end up with the one loop partition function for  $\mathcal{N} = 1$  supergravity as

$$Z = \prod_{k \geq 0} \frac{(1 + q^{\frac{3}{2}+k})^2}{(1 - q^{2+k})^2}, \quad (5.11)$$

which reproduces the result obtained by the heat kernel method in [2].

## 6. Concluding remarks

We have evaluated the one loop partition function for  $\mathcal{N} = 1$  supergravity in  $\text{AdS}_3$  from scratch. In passing, we have also provided an explicit expression of one loop determinant for a field of arbitrary spin in  $\text{AdS}_3$ . Although the relevant result is in good agreement with the previous one in the literature, the method we have employed here is not only essentially new but also amazingly powerful, which makes the whole calculation as simple as possible. For one thing, instead of the conventional Faddeev-Popov trick, we work out the determinant expression for one loop partition function of  $\mathcal{N} = 1$  supergravity by the decomposition technique, which simplifies the involved calculation very much. For another, with the recently discovered formula, the off shell one loop determinant can be remarkably expressed in terms of on shell quantities like normal modes, which turns out to be easy to construct in a purely algebraic way. In all, the whole procedure demonstrated here seems much simpler than the heat kernel method developed in [2], though there may be a close relation between them since the heat kernel is derived also by a group theoretical approach in [2].

We conclude with various generalizations of our work. First, taking into account that the higher spin supergravity has recently been proposed in [9], one is tempted to exploit the procedure developed here to address the analogous issue for this theory in  $\text{AdS}_3$ . Second, when the black hole is present, the normal modes will be replaced by the quasinormal modes. As shown in [6], the spectrum of quasinormal modes can also be constructed in a purely algebraic way, namely by the infinite tower of descendants of the chiral highest weight mode in the BTZ black hole. So it is highly interesting to see how the one loop determinant can be derived explicitly from quasinormal modes in the BTZ black hole, although it can also be obtained by a modular transformation from that in  $\text{AdS}_3$  on general grounds<sup>6</sup>. We hope to address these issues elsewhere in the near future.

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<sup>6</sup>It is noteworthy that such an investigation has been recently made for higher spin fields in [10] and [11], where nevertheless the relevant construction involves a little bit a posteriori arguments.

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## Appendices

### A. Some properties of Gamma matrices

By definition of Gamma matrices, we have

$$\begin{aligned}\gamma^a \gamma^b &= g^{ab} + \gamma^{ab}, \\ \gamma^a \gamma^b \gamma^c &= \gamma^{abc} + g^{ab} \gamma^c - g^{ca} \gamma^b + g^{bc} \gamma^a.\end{aligned}\tag{6.1}$$

On the other hand, special to the three dimension, we have

$$\gamma^{ab} = \epsilon^{abc} \gamma_c,\tag{6.2}$$

whereby we can further have

$$\epsilon_{abd} \gamma^{ab} = \epsilon_{abd} \epsilon^{abc} \gamma_c = -2\delta_d^c \gamma_c = -2\gamma_d,\tag{6.3}$$

and

$$\epsilon_{abd} \gamma^{ab} \gamma^d = \epsilon_{abd} \gamma^{abd} = -2\gamma_d \gamma^d = -6,\tag{6.4}$$

where the latter implies

$$\gamma^{abc} = \epsilon^{abc}.\tag{6.5}$$

In addition, we can also have

$$\gamma_{ab} \gamma^{ab} = \epsilon_{abc} \gamma^c \epsilon^{abd} \gamma_d = -2\delta_c^d \gamma^c \gamma_d = -2\gamma_d \gamma^d = -6.\tag{6.6}$$

### B. Explicit calculations for $Z_{ghost}$

Let us firstly define the path integral measures for scalar, vector, and tensor fields by requiring

$$\begin{aligned}\int \mathcal{D}\beta e^{-\langle\beta,\beta\rangle} &= 1, \\ \int \mathcal{D}\mu_a e^{-\langle\mu,\mu\rangle} &= 1, \\ \int \mathcal{D}\nu_{ab} e^{-\langle\nu,\nu\rangle} &= 1\end{aligned}\tag{6.7}$$

with

$$\begin{aligned}
\langle \beta, \beta' \rangle &= \int d^3x \sqrt{-g} \beta \beta', \\
\langle \mu, \mu' \rangle &= \int d^3x \sqrt{-g} \mu^a \mu'_a, \\
\langle \nu, \nu' \rangle &= \int d^3x \sqrt{-g} \nu^{ab} \nu'_{ab}.
\end{aligned} \tag{6.8}$$

Now decompose the vector field  $\xi^a$  as

$$\xi^a = \xi^{Ta} + \nabla^a \lambda \tag{6.9}$$

with  $\nabla_a \xi^{Ta} = 0$ , then the corresponding Jacobian can be obtained as follows

$$\begin{aligned}
1 &= \int \mathcal{D}\xi_a e^{-\langle \xi, \xi \rangle} = \int J_1 \mathcal{D}\xi_a^T \mathcal{D}\lambda e^{-\int d^3x \sqrt{-g} (\xi^{Ta} \xi_a^T - \lambda \nabla^a \nabla_a \lambda)} \\
&= J_1 [\det_0(-\nabla^a \nabla_a)]^{-\frac{1}{2}},
\end{aligned} \tag{6.10}$$

where as we have done throughout our paper any total derivative term is always thrown away. On the other hand, with the above decomposition, we have

$$h_{ab} = h_{ab}^T + \frac{1}{3} g_{ab} \alpha + \nabla_a \xi_b^T + \nabla_b \xi_a^T + 2 \nabla_a \nabla_b \lambda, \tag{6.11}$$

which gives rise to

$$\begin{aligned}
1 &= \int \mathcal{D}h_{ab} e^{-\langle h, h \rangle} \\
&= \int J_2 \mathcal{D}h_{ab}^T \mathcal{D}\alpha' \mathcal{D}\xi_a^T \mathcal{D}\lambda e^{-\int d^3x \sqrt{-g} [h^{Tab} h_{ab}^T + \frac{1}{3} \alpha'^2 + 2 \xi^{Ta} (-\nabla^b \nabla_b + 2) \xi_a^T + \frac{8}{3} \lambda (-\nabla^a \nabla_a + 3) (-\nabla^b \nabla_b) \lambda]} \\
&= J_2 [\det_1(-\nabla^a \nabla_a + 2)]^{-\frac{1}{2}} [\det_0(-\nabla^a \nabla_a + 3)]^{-\frac{1}{2}} [\det_0(-\nabla^a \nabla_a)]^{-\frac{1}{2}},
\end{aligned} \tag{6.12}$$

where  $\alpha' = \alpha + 2 \nabla^a \nabla_a \lambda$ . Combining (6.10) with (6.12) and taking into account  $\mathcal{D}\alpha' = \mathcal{D}\alpha$  at the same time, we end up with

$$Z_{ghost} = \frac{J_2}{J_1} = [\det_1(-\nabla^a \nabla_a + 2)]^{\frac{1}{2}} [\det_0(-\nabla^a \nabla_a + 3)]^{\frac{1}{2}}. \tag{6.13}$$

Similarly, let us define the path integral measures for fermion fields as

$$\begin{aligned}
\int \mathcal{D}\psi e^{-\langle \psi, \psi \rangle} &= 1, \\
\int \mathcal{D}\varsigma_a e^{-\langle \varsigma, \varsigma \rangle} &= 1
\end{aligned} \tag{6.14}$$



with

$$\begin{aligned}\langle \psi, \psi' \rangle &= \int d^3x \sqrt{-g} \bar{\psi} \psi', \\ \langle \varsigma, \varsigma' \rangle &= \int d^3x \sqrt{-g} \bar{\varsigma}^a \varsigma'_a.\end{aligned}\tag{6.15}$$

Then we have

$$\begin{aligned}1 &= \int \mathcal{D}\varphi_a e^{-\langle \varphi, \varphi \rangle} = \int Z_{ghost} \mathcal{D}\varphi_a^T \mathcal{D}\chi \mathcal{D}\zeta e^{-\langle \varphi, \varphi \rangle} \\ &= \int Z_{ghost} \mathcal{D}\varphi_a^T \mathcal{D}\chi' \mathcal{D}\zeta e^{-\int d^3x \sqrt{-g} [\bar{\varphi}_a^T \varphi^{Ta} - \frac{1}{3} \bar{\chi}' \chi' + (\nabla_a \bar{\zeta} - \frac{1}{3} \nabla_b \bar{\zeta} \gamma^b \gamma_a) (\nabla^a \zeta - \frac{1}{3} \gamma^a \gamma^c \nabla_c \zeta)]} \\ &= \int Z_{ghost} \mathcal{D}\zeta e^{-\int d^3x \sqrt{-g} \bar{\zeta} (-\nabla_a + \frac{1}{3} \gamma^b \gamma_a \nabla_b) (\nabla^a - \frac{1}{3} \gamma^a \gamma^c \nabla_c) \zeta} \\ &= \int Z_{ghost} \mathcal{D}\zeta e^{-\frac{2}{3} \int d^3x \sqrt{-g} \bar{\zeta} (-\nabla^a \nabla_a + \frac{3}{4}) \zeta} = Z_{ghost} \det_{\frac{1}{2}} (-\nabla^a \nabla_a + \frac{3}{4})\end{aligned}\tag{6.16}$$

where  $\chi' = \chi - 3\hat{m}\zeta + \gamma^a \nabla_a \zeta$ . So we end up with

$$Z_{ghost} = [\det_{\frac{1}{2}} (-\nabla^a \nabla_a + \frac{3}{4})]^{-1}.\tag{6.17}$$

### C. Proof of $\det(\gamma^a \nabla_a + \hat{m}) = \det(\gamma^a \nabla_a - \hat{m})$

In order to prove the above identity, we are only required to show the eigenvalues of  $\gamma^a \nabla_a$  always come in pairs as  $\{C, -C\}$ , which can be demonstrated by acting on  $\varphi_b$  in the following way. Namely assume  $\gamma^a \nabla_a \varphi_b(t, \phi, \rho) = C \varphi_b(t, \phi, \rho)$  and  $\varphi'_b(t, \phi, \rho) = \varphi_0(t, -\phi, \rho)(dt)_b - \varphi_1(t, -\phi, \rho)(d\phi)_b + \varphi_2(t, -\phi, \rho)(d\rho)_b = \varphi_b(t, \phi', \rho)|_{\phi' = -\phi}$ , then we have

$$\begin{aligned}\gamma^a \nabla_a \gamma^1 \varphi'_b(t, \phi, \rho) &= \gamma^a \gamma^1 \partial_a \varphi'_c(t, \phi, \rho) \\ &\quad + \frac{1}{4} \gamma^a \omega_{IJa} \gamma^{IJ} \gamma^1 \varphi'_c(t, \phi, \rho) + \gamma^a \Gamma^c_{ab} \gamma^1 \varphi'_c(t, \phi, \rho) \\ &= -\gamma^1 \gamma^a \nabla_a \varphi_b(t, \phi', \rho)|_{\phi' = -\phi} = -C \gamma^1 \varphi_b(t, \phi', \rho)|_{\phi' = -\phi} \\ &= -C \gamma^1 \varphi'_b(t, \phi, \rho),\end{aligned}\tag{6.18}$$

where we have used the fact that the non-vanishing Christoffel symbols as well as non-vanishing spin connections are given by

$$\begin{aligned}\Gamma^0_{02} &= \frac{\sinh(\rho)}{\cosh(\rho)}, \Gamma^1_{12} = \frac{\cosh(\rho)}{\sinh(\rho)}, \Gamma^2_{00} = \sinh(\rho) \cosh(\rho), \Gamma^2_{11} = -\sinh(\rho) \cosh(\rho), \\ \omega_{02a} &= -\sinh(\rho)(dt)_a, \omega_{12a} = \cosh(\rho)(d\phi)_a\end{aligned}\tag{6.19}$$

with the choice of the orthogonal normal bases as  $\{e_0^a = \frac{1}{\cosh(\rho)}(\frac{\partial}{\partial t})^a, e_1^a = \frac{1}{\sinh(\rho)}(\frac{\partial}{\partial \phi})^a, e_2^a = (\frac{\partial}{\partial \rho})^a\}$  in the coordinate system  $\{t, \phi, \rho\}$ .

So we have accomplished our proof by showing that  $\gamma^1 \varphi'_b$  is the eigenvector with eigenvalue  $-C$  if  $\varphi_b$  is the eigenvector with eigenvalue  $C^7$ .

## D. Some properties of Killing vector fields

A Killing vector field  $\xi^a$ , by definition, is a vector field satisfying

$$\mathcal{L}_\xi g_{ab} = \nabla_a \xi_b + \nabla_b \xi_a = 2\nabla_{(a} \xi_{b)} = 0. \quad (6.20)$$

Whence one can show such a Killing vector field also has the following nice properties, i.e.,

$$\begin{aligned} \mathcal{L}_\xi \epsilon &= 0, \\ \mathcal{L}_\xi \nabla_a &= \nabla_a \mathcal{L}_\xi, \\ \mathcal{L}_\xi \gamma^a &= \gamma^a \mathcal{L}_\xi. \end{aligned} \quad (6.21)$$

Now let us prove the first one by working on the three dimensional case as follows

$$\mathcal{L}_\xi \epsilon_{abc} = \xi^d \nabla_d \epsilon_{abc} + \epsilon_{dbc} \nabla_a \xi^d + \epsilon_{adc} \nabla_b \xi^d + \epsilon_{abd} \nabla_c \xi^d = \epsilon_{dbc} \nabla_a \xi^d + \epsilon_{adc} \nabla_b \xi^d + \epsilon_{abd} \nabla_c \xi^d. \quad (6.22)$$

To achieve our proof, we only need to show that the quantity  $\epsilon^{abc} \mathcal{L}_\xi \epsilon_{abc}$  vanishes. This is the case because

$$\epsilon^{abc} (\epsilon_{dbc} \nabla_a \xi^d + \epsilon_{adc} \nabla_b \xi^d + \epsilon_{abd} \nabla_c \xi^d) = -6 \nabla_d \xi^d = 0. \quad (6.23)$$

Next let us demonstrate why the Lie derivative via Killing vector fields commutes with both the covariant derivative and gamma matrices by acting on the spinor field in the following way, i.e.,

$$\begin{aligned} \mathcal{L}_\xi \nabla_a \psi - \nabla_a \mathcal{L}_\xi \psi &= \xi^b \nabla_b \nabla_a \psi + \nabla_b \psi \nabla_a \xi^b - \frac{1}{4} \gamma^{cd} \nabla_a \psi \nabla_d \xi_c - \nabla_a (\xi^b \nabla_b \psi - \frac{1}{4} \gamma^{cd} \psi \nabla_d \xi_c) \\ &= \xi^b (\nabla_b \nabla_a - \nabla_a \nabla_b) \psi + \frac{1}{4} \gamma^{cd} \psi \nabla_a \nabla_d \xi_c \\ &= \frac{1}{4} \xi^b R_{bacd} \gamma^{cd} \psi + \frac{1}{4} \gamma^{cd} \psi R_{cdab} \xi^b \\ &= \frac{1}{4} \xi^b R_{bacd} \gamma^{cd} \psi + \frac{1}{4} R_{abcd} \xi^b \gamma^{cd} \psi = 0, \end{aligned} \quad (6.24)$$

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<sup>7</sup>Actually this can be regarded as the analogous version of parity operation in AdS<sub>3</sub> background.

and

$$\begin{aligned}
\mathcal{L}_\xi \gamma^a \psi - \gamma^a \mathcal{L}_\xi \psi &= \xi^b \nabla_b (\gamma^a \psi) - \gamma^b \psi \nabla_b \xi^a - \frac{1}{4} \gamma^{cd} \gamma^a \psi \nabla_d \xi_c - \gamma^a (\xi^b \nabla_b \psi - \frac{1}{4} \gamma^{cd} \psi \nabla_d \xi_c) \\
&= -\gamma^b \psi \nabla_b \xi^a - \frac{1}{4} (\gamma^{cd} \gamma^a - \gamma^a \gamma^{cd}) \psi \nabla_d \xi_c = -\gamma^b \psi \nabla_b \xi^a + g^{ac} \gamma^d \psi \nabla_d \xi_c \\
&= -\gamma^b \psi \nabla_b \xi^a + g^{ac} \gamma^d \psi \nabla_d \xi_c = 0.
\end{aligned} \tag{6.25}$$

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